Surfaces of bifurcation in a triparametric quadratic Hamiltonian

V. Lanchares, M. Iñarrea, J.P. Salas, and J.D. Sierra
Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain

A. Elipe
Grupo de Mecánica Espacial, Universidad de Zaragoza, 50009 Zaragoza, Spain
(Received 5 April 1995)

Numerous dynamical systems are represented by a quadratic Hamiltonian with the phase space on the S^2 sphere. In this paper, for the unique class of quadratic Hamiltonians depending on three parameters, we analyze the equilibria and the occurrence of parametric bifurcations; we obtain the surfaces of bifurcation in the space of parameters. We describe, in the context of quadratic Hamiltonians, a special type of bifurcation associated with a nonelementary fixed point; we name it a *double teardrop* bifurcation.

PACS number(s): 03.20.+i, 03.65.Sq, 46.10.+z

I. INTRODUCTION

Perturbation methods have brought new insights in the analysis of nonlinear dynamical Hamiltonian systems [1]. By means of averaging or normalization techniques, the original nonintegrable Hamiltonian is replaced by an integrable approximation that is built so as to give good agreement with the real dynamics.

In a large number of cases the resulting Hamiltonian proves to be quadratic in a set of variables that generate an su(2) algebraic structure. In these variables the system has a singular noncanonical bracket tensor, and a conserved quantity, called the *Casimir* invariant, exists due to the degeneracy of the bracket [2,3]. The topology of the phase space is defined by a constant-energy sphere (Hopf sphere).

We find this type of problem in classical mechanics, e.g., in the rigid body in torque free motion [4–6], the motion of atoms in electric and magnetic fields [1,7–9], in studies on molecules [10], in the context of Hamiltonian chaos in nonlinear optical polarization dynamics [11,12], in the detection of frozen orbits in the artificial satellite theory [13,14], and in galactic dynamics [15], among others.

In most of the cases previously mentioned, parametric bifurcations have been detected by numerical analysis for concrete values of the parameters of the system, which means that most of the dynamics of the system is uncovered. These facts motivated several authors to make a classification depending on the parameters. Indeed, when such a classification is available, together with the different bifurcations occurring in each class, it becomes easy, for a given parametric quadratic Hamiltonian, to identify the bifurcations inherent to it. For biparametric quadratic Hamiltonians, a complete classification has already been done in [16], and the parametric bifurcations and global phase flow representation is contained in [17,18]. For more than two parameters, the classification of the types of Hamiltonian is given in [19].

In this paper, we deal with the unique triparametric quadratic Hamiltonian. This Hamiltonian is equivalent to case 5 in the work of David *et al.* [12, p. 348], corresponding to the one-beam problem with birefringence and that was left unanalyzed because of its complexity [12, p. 364]; thus, new bifurcations might be expected, and indeed, this is the case.

From this triparametric quadratic Hamiltonian, we derive the equations of the motion and the equilibria (Sec. II) and the regions in the parametric space where they exist. The boundaries of these regions (a cone and a conoid) determine the parametric bifurcations (Sec. III). The stability of the equilibria in each region is obtained (Sec. IV), and the phase flow evolution is described in Sec. V. The biparametric case [11,17,18] is recovered as a particular case of the triparametric one.

II. HAMILTONIAN AND EQUILIBRIA

Let us consider a Hamiltonian system with one degree of freedom in a set of variables u, v, w such that they generate an su(2) algebraic structure, that is, their Poisson brackets satisfy the relations

$$(u;v) = w, \qquad (v;w) = u, \qquad (w;u) = v. \tag{1}$$

According to these relations, the variables (u, v, w) lie on a two-dimensional sphere S^2. For the sake of simplifying the notation we assume the radius of the sphere is equal to unity, that is,

$$u^2 + v^2 + w^2 = 1. \tag{2}$$

Once the symplectic structure is defined, we shall consider dynamical systems represented by a Hamiltonian that is a quadratic form in these spherical coordinates. Its general expression is

1063-651X/95/52(5)/5540(9)/$06.00

$$\mathcal{H} = \frac{1}{2}Au^2 + \frac{1}{2}Bv^2 + \frac{1}{2}Cw^2 + Duv + Euw + Fvw + Lu + Mv + Nw,$$

where the coefficients are real parameters independent of the variables. Under an action of the SO(3) group (see [16]) the Hamiltonian is reduced to

$$\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Bv^2 + \frac{1}{2}Cw^2 + Lu + Mv + Nw. \quad (3)$$

A complete classification of this type of Hamiltonian system in terms of the number of free parameters is made in [19].

The global phase flow and its bifurcations for the biparametric cases have been studied in [17,18]. In the present paper we focus our attention on Hamiltonians of the type (3) that depend on three parameters. As was established in [19], there is only one triparametric case:

$$\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}Pv^2 + Qu + Rv. \quad (4)$$

The presence of symmetries simplifies the analysis of the phase portrait. The Hamiltonian $\mathcal{H}$ is invariant under the transformations

$$\begin{aligned} w &\longrightarrow -w, \\ (u,Q) &\longrightarrow (-u,-Q), \\ (v,R) &\longrightarrow (-v,-R). \end{aligned} \quad (5)$$

The first symmetry shows that the phase flow is symmetric with respect to the plane $w = 0$, and the other two show that it is sufficient to study nonnegative values of the parameters Q and R.

Taking into account the Liouville-Jacobi theorem and the relations (1), the equations of the motion are

$$\begin{aligned} \dot{u} &= (u;\mathcal{H}) = w(Pv + R), \\ \dot{v} &= (v;\mathcal{H}) = -w(u + Q), \\ \dot{w} &= (w;\mathcal{H}) = v(u + Q) - u(Pv + R). \end{aligned} \quad (6)$$

These equations present the symmetries

$$\begin{aligned} (w,t) &\longrightarrow (-w,-t), \\ (u,Q,t) &\longrightarrow (-u,-Q,-t), \\ (v,R,t) &\longrightarrow (-v,-R,-t), \end{aligned}$$

which correspond to the symmetries (5).

In some particular cases, the system (6) is easily integrable. Indeed, when $P = 0$ and $R = 0$, the right hand member of the first equation vanishes and the motion is made of pure rotations around the u axis with constant angular velocity $(Q + u)$. In the case that $P = 1$, $Q = 0$, and $R = 0$, the situation is analogous, but now the rotations are around the w axis with constant angular velocity w.

Equilibria are one of the main sources of information on the phase flow. The equilibria are the roots of the system made of the right hand members of (6) equaled to zero, together with the constraint (2).

For some particular values of the parameters, the system presents degeneracies, that is, the set of equilibria is dense, or in other words, there are nonisolated equilibria. The parameters for which this happens are the above mentioned. Indeed, (i) for $P = 0$, $R = 0$, and $|Q| \leq 1$, all points on the small circle $v^2 + w^2 = 1 - Q^2$ are equilibria, and (ii) for $P = 1$, $Q = 0$, and $R = 0$ the equator ($w = 0$) of S^2 is made of equilibria. The isolated equilibria are obtained depending on the value of w.

A. $w \neq 0$

The first equation of the system (6) vanishes for $v = -R/P$, the second one when $u = -Q$, and for these values of u and v, the third equation vanishes too. Taking into account that the points must lie on the sphere (2), we find two equilibria, namely,

$$E_0 \equiv \left(-Q, -R/P, \sqrt{1 - Q^2 - R^2/P^2}\right)$$

and

$$E_1 \equiv \left(-Q, -R/P, -\sqrt{1 - Q^2 - R^2/P^2}\right).$$

N.B. For the existence of these points the condition $Q^2 + R^2/P^2 \leq 1$ must hold, i.e., these equilibria only exist in the interior of the set

$$\mathcal{F} \equiv \left\{(P,Q,R) \,\middle|\, R^2 \leq P^2(1 - Q^2)\right\}. \quad (7)$$

In the boundary $\partial\mathcal{F}$, the two points E_0, E_1 merge into only one, $(-Q, -R/P, 0)$. Consequently, $\partial\mathcal{F}$ defines a surface of parametric bifurcation in the parameter space. Such a surface (Fig. 1) is not differentiable along the segment $P = R = 0$, $|Q| \leq 1$, that coincides precisely with the first degenerate case above mentioned.

This surface is a ruled surface, and is named a *conoid* [20, p. 194]; its level contours on the plane $P =$ const are ellipses, one of its semiaxes of constant length equal to the unit, and the other of variable length (P). The

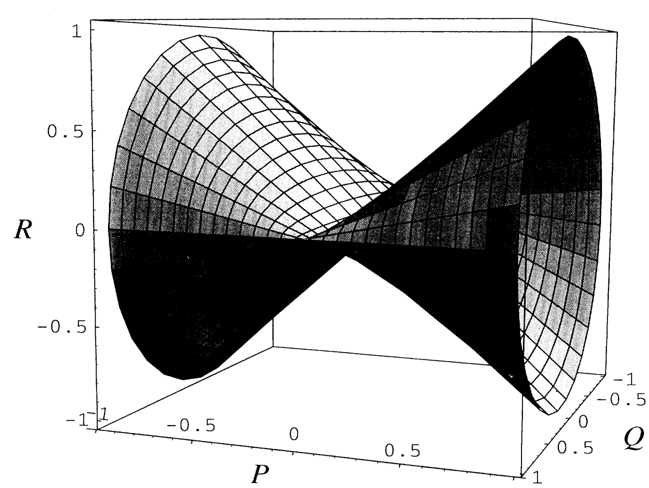

FIG. 1. Surface of bifurcation $\partial\mathcal{F} \equiv R^2 = P^2(1 - Q^2)$ for equilibria with $w \neq 0$.

level contours on the plane Q = const are straight lines, passing through the point $P = 1$. The intersection of the surface with the plane $R = 0$ are two parallel lines $Q = \pm 1$.

B. $w = 0$

Now, the first two equations of (6) are equal to zero and the third one vanishes in the following two cases:

$$u = 0, \quad v = \pm 1 \qquad \text{when} \quad P = 1, \quad Q = 0, \tag{8}$$

and

$$v = \frac{uR}{(1-P)u + Q} \qquad \text{when} \quad P \neq 1, \quad Q \neq 0. \tag{9}$$

Let us consider now the relation (9). In this case, and remembering that the variables are on the sphere (2), it follows that the first component of the equilibria (u) must fulfill the following polynomial equation:

$$\mathcal{A}(u) = (u^2 - 1)\,[(1-P)u + Q]^2 + R^2u^2 = 0. \tag{10}$$

Between the roots of the polynomial $\mathcal{A}$ and the equilibria there is a one-to-one correspondence, for $\mathcal{A}$ is > 0 iff $|u| > 1$. By virtue of the Bolzano theorem [21, p. 96], there exist, at least, two equilibria in the plane $w = 0$, since $\mathcal{A}(-1) = R^2 > 0$, $\mathcal{A}(0) = -Q^2 < 0$, and $\mathcal{A}(1) = R^2 > 0$.

The polynomial $\mathcal{A}$ was obtained from (9), that is, for $P \neq 1$. Thus, the transformation $(Q^\star, R^\star) \longmapsto (Q, R)$ defined by

$$Q^\star = Q/(1-P), \qquad R^\star = R/(1-P) \tag{11}$$

is regular and converts $\mathcal{A}$ into

$$\mathcal{A}(u) = (u^2 - 1)(u + Q^\star)^2 + R^{\star 2}u^2 = 0. \tag{12}$$

Equation (12) coincides exactly with Eq. (8) of [17], which determines the equilibria for the biparametric quadratic Hamiltonian ($\mathcal{H} = \frac{1}{2}u^2 + Q^\star u + R^\star v$).

Let us recall that in this biparametric case, the parametric bifurcation line was characterized as the curve on which the equilibria were *nonelementary* fixed points, that is, points whose Poincaré index is zero. This type of singular point may be considered to be the coalescence of a center and a saddle point. The singularity [22,23] corresponds to an inflection point of the potential energy and the singular point must be regarded as unstable. The bifurcation line is the hypocycloid of four cusps,

$$Q^{\star 2/3} + R^{\star 2/3} = 1,$$

that by means of the inverse transformation of (11) is converted into the surface

$$\partial\mathcal{G} \equiv Q^{2/3} + R^{2/3} = (1-P)^{2/3}, \tag{13}$$

that is a cone (Fig. 2), with its vertex at the point $Q = 0, P = 1, R = 0$. The intersection with this cone on the plane P = const is an hypocycloid of four cusps, whereas the intersection with planes Q = const or R = const are two straight segments. Obviously, the cross of the cone $\mathcal{G}$ with the coordinate plane $R = 0$ are two straight lines passing through the vertex $Q = 0, P = 1, R = 0$.

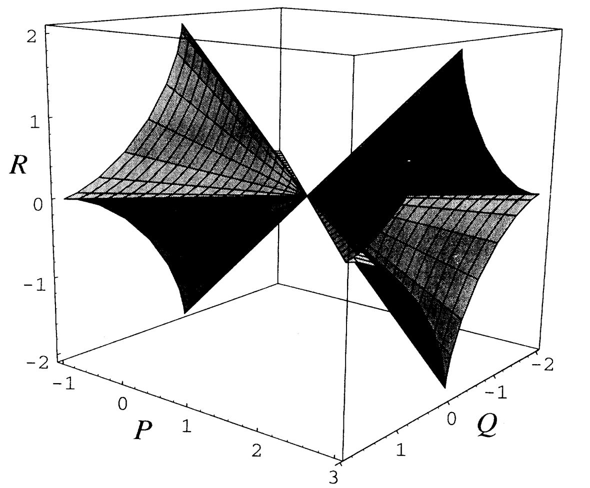

FIG. 2. Surface of bifurcation $\partial\mathcal{G} \equiv Q^{2/3} + R^{2/3} = (P-1)^{2/3}$ for equilibria with $w = 0$.

In the exterior of the set $\mathcal{G} \equiv \{(P, Q, R) \mid Q^{2/3} + R^{2/3} \leq (1-P)^{2/3}\}$, there are only two equilibria lying on the equator $w = 0$; in the interior of $\mathcal{G}$ there are four such points; and just on the boundary $\partial\mathcal{G}$, there are three equilibria with $w = 0$.

N.B. Setting $P = 0$ in the surfaces (7) and (13), we recover the bifurcation lines of the biparametric Hamiltonian studied in [17]; analogously, setting $R = 0$, we recover the bifurcation lines corresponding to the other biparametric Hamiltonian studied in [18].

III. PARAMETRIC BIFURCATIONS

From the analysis above made, for a given value of parameters Q, P, and R, the phase flow will depend on the relative position of this point (in the QPR space of parameters) with respect to the two surfaces $\partial\mathcal{F}$ and $\partial\mathcal{G}$. The existence of the symmetries (5) allows us to restrict the analysis to values non-negative of the parameters Q and R.

To study the crossing of the two surfaces $\partial\mathcal{F}$ and $\partial\mathcal{G}$, we take the sections of them with planes $P = P_0$ = const. The section of the conoid $\partial\mathcal{F}$ with such a plane is the ellipse

$$Q^2 + \frac{R^2}{P_0^2} = 1, \tag{14}$$

whose semiaxes are 1 and P_0. On the other hand, the section of the cone $\partial\mathcal{G}$ is the hypocycloid of four cusps (astroid),

$$Q^{2/3} + R^{2/3} = (1 - P_0)^{2/3}. \qquad (15)$$

Since we consider only the first quadrant $(Q, R > 0)$, the ellipse (14) is concave (i.e., $d^2R/dQ^2 < 0$), whereas the astroid (15) is convex (i.e., $d^2R/dQ^2 > 0$). Both curves decrease continuously as functions of Q.

The ellipse cuts the coordinate axes at the points $(Q, R) = (0, |P|)$ and $(Q, R) = (1, 0)$; the astroid cuts it at the points $(Q, R) = (0, |1 - P|)$ and $(Q, R) = (|1 - P|, 0)$. Since for

$$P_0 \in (0, \tfrac{1}{2}) \Longrightarrow 1 - P > P \quad \text{and} \quad 1 - P < 1,$$

$$P_0 \in (2, \infty) \Longrightarrow |1 - P| < P \quad \text{and} \quad |1 - P| > 1,$$

it follows that in these intervals the curves intersect one another once.

However, for

$$P_0 \in (\tfrac{1}{2}, 2) \Longrightarrow |1 - P| < P \quad \text{and} \quad |1 - P| < 1,$$

and because of the different convexity, the curves do not intersect.

For

$$P_0 \in (-\infty, 0) \Longrightarrow |1 - P| > |P| \quad \text{and} \quad |1 - P| > 1,$$

and it follows that either the curves do not intersect or they cross one another twice. However, from the plots of the surfaces, it seems that both surfaces (for $P < 0$) are tangent, that is, the two crossing points merge into only one. Let us prove that this is the case.

Different representations for the ellipse (14) and the hypocycloid (15) are

$$\text{ellipse}: \quad \sigma \equiv (|1 - P_0| \sin^3 \tau, |1 - P_0| \cos^3 \tau), \quad \text{with} \quad 0 \le \tau \le \pi/2, \qquad (16)$$

$$\text{hypocycloid}: \quad \Sigma \equiv (\sin \tau, |P_0| \cos \tau), \quad \text{with} \quad 0 \le \tau \le \pi/2. \qquad (17)$$

The tangent vectors to these curves are, respectively,

$$\sigma' = (3|1 - P_0| \sin^2 \tau \cos \tau, -3|1 - P_0| \cos^2 \tau \sin \tau)$$

and

$$\Sigma' = (\cos \tau, -|P_0| \sin \tau).$$

Let us see under what conditions these vectors are parallel. In this case,

$$3|1 - P_0| \sin^2 \tau \cos \tau = \alpha \cos \tau$$

and

$$-3|1 - P_0| \cos^2 \tau \sin \tau = -\alpha |P_0| \sin \tau,$$

with α a parameter, that is,

$$3|1 - P_0| \sin^2 \tau = \alpha \quad \text{and} \quad 3|1 - P_0| \cos^2 \tau = \alpha |P_0|,$$

that when substituted into (16) and (17) give the points on the ellipse and on the hypocycloid for which the tangent vectors are parallel:

$$\sigma(P_0) \equiv \left((1 - P_0) \left[\frac{\alpha}{3(1 - P_0)} \right]^{3/2}, \ (1 - P_0) \left[\frac{\alpha P_0}{3(1 - P_0)} \right]^{3/2} \right)$$

and

$$\Sigma(P_0) \equiv \left(\left[\frac{\alpha}{3(1 - P_0)} \right]^{1/2}, \ P_0 \left[\frac{\alpha P_0}{3(1 - P_0)} \right]^{1/2} \right),$$

and both points coincide when $\alpha = 3$. The common point is obtained for a value of τ given by

$$\sin^2 \tau = \frac{1}{|1 - P_0|}, \qquad \cos^2 \tau = \frac{|P_0|}{|1 - P_0|},$$

and these relations only hold for $P_0 < 0$. That is, when $P < 0$, the conoid (7) and the cone (13) are tangent to one another. For $1/2 < P < 2$, the surfaces do not intersect (the cone is inside the conoid) (Fig. 3).

It turns out that the division of the space of parameters (Q, P, R) can be visualized by means of four cuts across the P axis as it is depicted in Fig. 4; the regions are labeled, in the figure, with capital letters and the primes stand for those regions with a qualitatively equivalent phase flow. Besides, we have to account for the loci where there are nonisolated equilibria, namely, the point $P = 1$, $Q = 0$, $R = 0$ [the vertex of the cone (13)] and the segment $P = 0$, $R = 0$, $|Q| \le 1$ [the edge of the surface (7)].

IV. STABILITY

Explicit linear stability boundaries for equilibria of a one-degree-of-freedom Hamiltonian flow are determined by the characteristic polynomial. Since eigenvalues occur in positive and negative pairs $\pm\lambda$, it is possible to reduce the degree of the characteristic polynomial by a factor of 2 [24]. Therefore, a reduced characteristic polynomial of the form $\mathcal{Q}(\sigma) = \sigma - A$ is obtained and the equilibrium

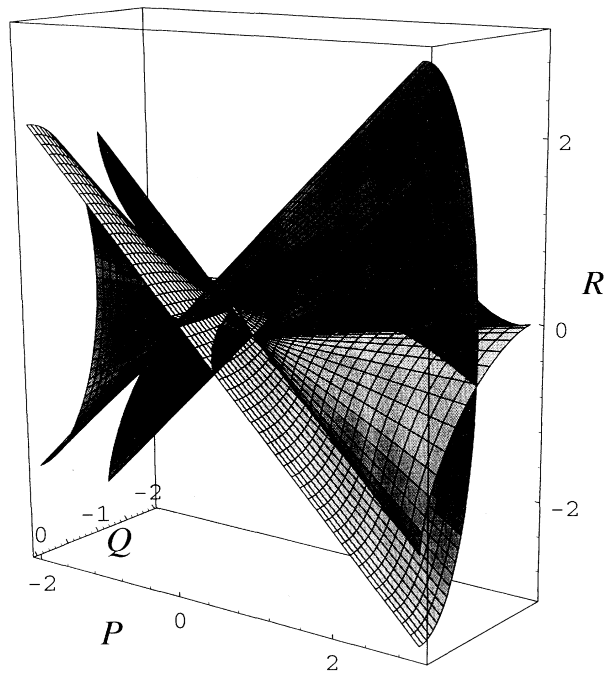

FIG. 3. The two surfaces of parametric bifurcations: the conoid $\partial\mathcal{F}$ and the cone $\partial\mathcal{G}$.

is stable if $A \geq 0$ with stability boundary at $A = 0$.

All the above indicates that determining the stability of the equilibria of a one-degree-of-freedom Hamiltonian flow is not a difficult task. However, in the previous statement, the coordinates of the equilibrium are supposed to be known, whereas in the present paper we deal with equilibria for which we do not know explicitly their coordinates. Nevertheless, we will show that it is possible to determine the stability of the fixed points albeit their coordinates are not known.

To begin with, we linearize the equations of the motion (6), from which we obtain the characteristic polynomial

$$\begin{aligned}\lambda^3 + &\Big[(u+Q)(u+Q-Pu)\\ &+(Pv+R)(Pv-v+R)+Pw^2\Big]\lambda\\ &+\Big[(Pv+R)(u+Q-Pu)\\ &+(u+Q)(v-Pv-R)P\Big]w. \end{aligned} \tag{18}$$

Note that, contrary to what was mentioned previously, eigenvalues do not occur in pairs. This is due to the *Casimir function* [2,3] $C = u^2+v^2+w^2$, which introduces a noncanonical formalism and, consequently, the phase flow is lying on the sphere S^2.

A. Stability of E_0 and E_1 ($w \neq 0$)

The equilibria E_0 and E_1 only exist in the interior of $\mathcal{F}$, that is, when $R^2 < P^2(1-Q^2)$. For these points, the characteristic polynomial (18) becomes

$$\lambda^3 + Pw^2\lambda = \lambda^3 + \left(1 - Q^2 - \frac{R^2}{P^2}\right)P\lambda.$$

The eigenvalue $\lambda = 0$ is due to the condition

$$u\delta u + v\delta v + w\delta w = 0$$

resulting from the variation of the Casimir. Thus, the reduced characteristic polynomial is

$$\lambda^2 + Pw^2 = \lambda^2 + \left(1 - Q^2 - \frac{R^2}{P^2}\right)P.$$

Taking into account that w^2 is always positive, we find that E_0 and E_1 are stable iff $P > 0$, and E_0 and E_1 are unstable iff $P < 0$.

B. Stability of the equilibria in the plane $w = 0$

The equilibria placed on the plane $w = 0$ are given by Eqs. (9) and (10). By substituting (9) into (18), there results

$$\lambda^2 + (Q+u)\frac{QR^2 + [u(1-P)+Q]^3}{[u(1-P)+Q]^2}. \tag{19}$$

Thus, the linear stability boundaries are determined by the two relations

$$u = -Q$$

and

$$u = u_0 = -Q^{1/3}\frac{Q^{2/3}+R^{2/3}}{1-P},$$

and, besides, u still must fulfill (10); that is to say, when either

$$\mathcal{A}(-Q) = Q^2\left[R^2 + (Q^2-1)P^2\right] = 0$$

or

$$\mathcal{A}(u_0) = \frac{Q^{2/3}R^{4/3}}{(1-P)^2}\left[(Q^{2/3}+R^{2/3})^3 - (1-P)^3\right] = 0.$$

It is worth noting that the stability boundaries are, precisely, the surfaces $\partial\mathcal{F}$ and $\partial\mathcal{G}$ defined above. Therefore, the stability of an equilibrium in the plane $w = 0$ remains unchanged in each region in which the space of parameters is divided (see Fig. 4).

Let us analyze, for instance, the stability in the region C, that is, the intersection of the sets $\mathcal{F}$ and $\mathcal{G}$ with $P > 1$. For a such a point, we have $-1 < -Q < -Q/(1-P) < u_0 < 1$ and evaluating polynomial $\mathcal{A}$ there result

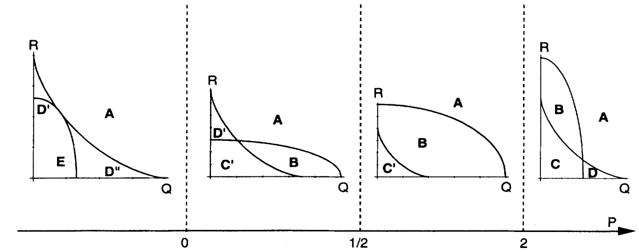

FIG. 4. Division of the space of parameters for different cuts across the P axis.

$$
\begin{aligned}
&\mathcal{A}(-1) = R^2 > 0,\\
&\mathcal{A}(-Q) = Q^2\left[R^2 + (Q^2-1)P^2\right] < 0,\\
&\mathcal{A}(-Q/(1-P)) = \tfrac{R^2Q^2}{(1-P)^2} > 0,\\
&\mathcal{A}(u_0) = \tfrac{Q^{2/3}R^{4/3}}{(1-P)^2}\left[(Q^{2/3}+R^{2/3})^3 - (1-P)^2\right] < 0,\\
&\mathcal{A}(1) = R^2 > 0,
\end{aligned}
$$

which means that for $u \in [-1,1]$ there are four equilibria. From Eq. (19), it is easily checked that the fixed points such that $u \in [-Q, -Q/(1-P)]$ and $u \in [-Q/(1-P), u_0]$ are stable, whereas $u \in [-1,-Q]$ and $u \in [u_0, 1]$ are unstable ones.

The stability of the other cases is obtained in a similar way. The stability-instability transition is depicted in Fig. 5.

V. GLOBAL PHASE PORTRAIT

Since the Hamiltonian (4) has only one degree of freedom, it is integrable, and its trajectories are the curves obtained as the intersection of the two surfaces, the Hamiltonian $\mathcal{H}$ and the sphere S^2, that is to say, the trajectories are the contour levels of the Hamiltonian on the sphere. To visualize the evolution of the phase flow when the parameters evolve, we will travel through the different regions and their boundaries along different paths.

There are several possible paths threading the regions in the space of parameters; from these, we choose those describing most of the bifurcations and for them, we draw the phase portrait (see Figs. 6–8). We find that a *pitchfork* bifurcation occurs when the surface (7) is crossed and a *teardrop* bifurcation when (13) is traversed. Basically these two bifurcations govern the transition from one region to another.

In addition to these cases, worthy regions for study are those where nonisolated equilibria exist, and also, the coordinate planes, which divide regions where the phase flow configuration is topologically equivalent due to the existing symmetries. These cases were already studied [17,18]; the bifurcations are of the types *butterfly* and *oyster*, therein described, and the reader is addressed to these references for details.

Besides, in the region $P < 0$ (see Fig. 8), where the two surfaces (7) and (13) are tangent to each another, we describe in the context of quadratic Hamiltonians on the sphere, a special type of bifurcation associated with a nonelementary fixed point. Next, we describe in detail the bifurcations occurring in that hemispace. (N.B. In planar dynamical systems, this bifurcation has been already found; see, for instance, the book of Guckenheimer and Holmes [25, p. 385]).

A. $P < 0$. Bifurcations through the four regions

To analyze in this case how the phase portrait evolves, let us move in the space of parameters along the path Γ, a circle around the tangent point of the surfaces (7) and (13) in the plane $P < 0$, for instance $P = -2$. It crosses the regions A, D′, E and D″. The phase flow evolution along the path is depicted in Fig. 8.

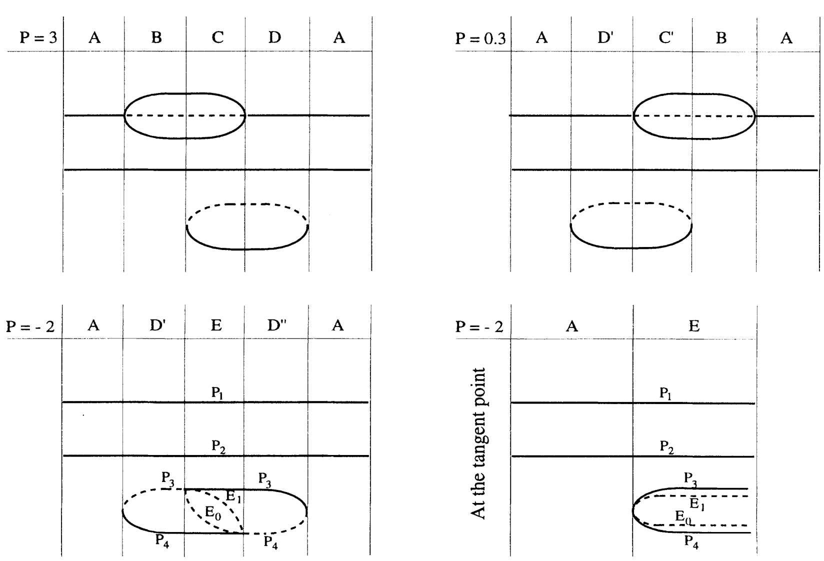

FIG. 5. Stability in the different regions. Continuous lines stand for stable points and dashed lines for unstable equilibria.

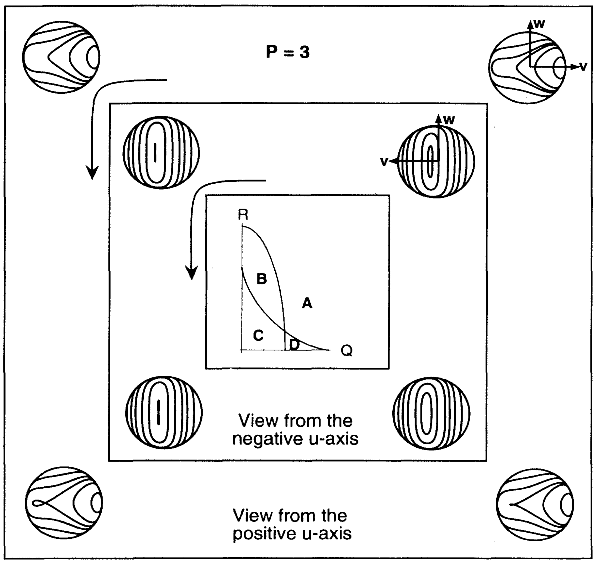

FIG. 6. Evolution of the phase flow for $P = 3$.

In region A, the flow consists of rotations around two stable points (P_1 and P_2) on the plane $w = 0$. When the path crosses the cone (13) and comes in region D′ a *teardrop* bifurcation takes place: one of the periodic orbits becomes an homoclinic loop that emanates from a degenerate equilibrium, P_3 (with Poincaré index 0) on the equator of the sphere. Once inside region D′, another homoclinic orbit emanates from the new equilibrium, now unstable, surrounding a new stable fixed point P_4, located on the plane $w = 0$. As the path moves in towards the region E the homoclinic loops tend to be tangent at

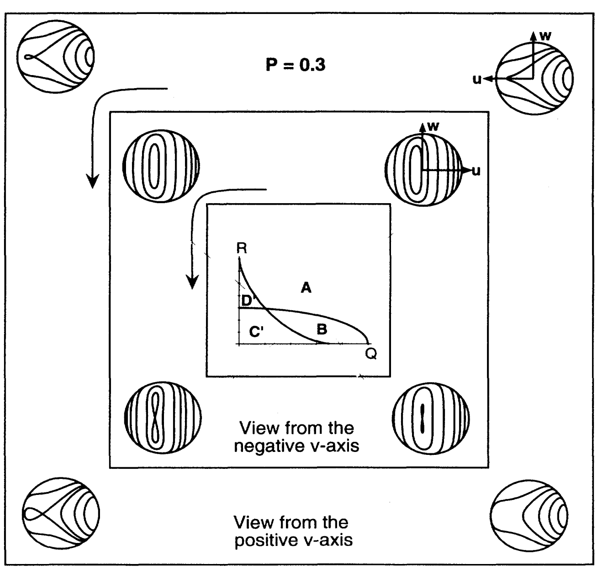

FIG. 7. Evolution of the phase flow for $P = 0.3$.

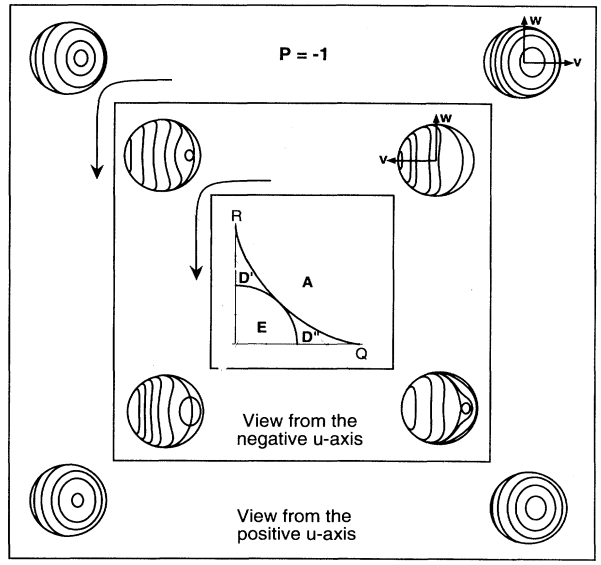

FIG. 8. Evolution of the phase flow through the path Γ for $P = -2$.

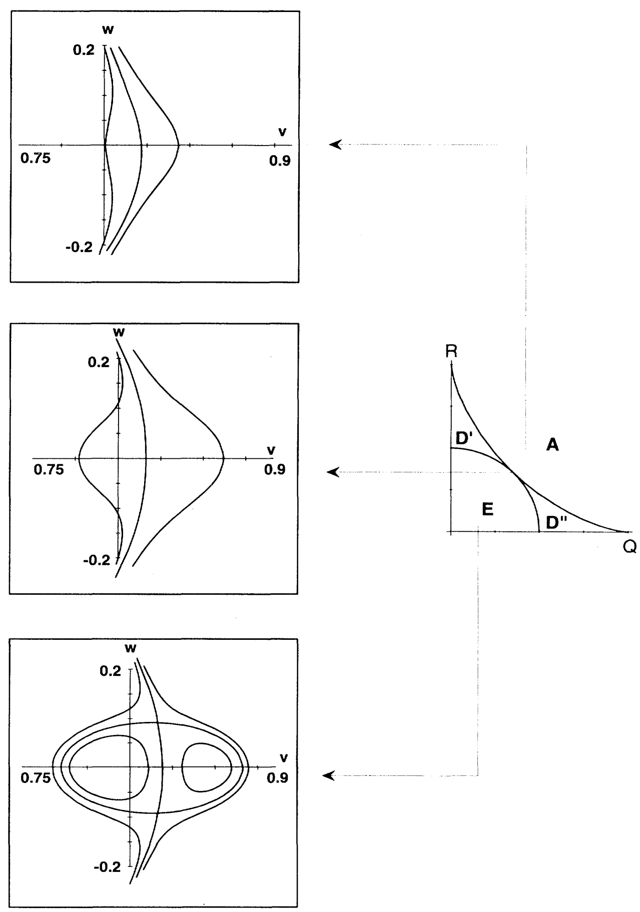

FIG. 9. Evolution of the phase flow through the common point of tangency between the two surfaces, the cone and he conoid, for $P = -2$. A *double teardrop* bifurcation appears at this point.

P_3, and the two loops of the figure-eight shape are tangent exactly at the conoid (7). Once inside this surface, the unstable point P_3 splits into three points, two unstable (E_0 and E_1) along the meridian (and symmetric with respect to the equator) and P_3 that now becomes stable. Although it may appear as a new kind of bifurcation, we recognize in it a pitchfork bifurcation.

As we move through region E along the path Γ, the homoclinic orbit surrounding P_4 abates, and again, at the conoid, the three equilibria E_0, E_5, and P_4 merge at only one point (P_4), and the two homoclinic surrounding P_1 and P_4, respectively, are tangent. Once the conoid is traversed, the figure-eight shape appears, and as we approach the cone, the loop containing P_3 diminishes its size, and vanishes exactly at the cone. At this surface, P_3 and P_4 merge into only one point (P_4) with Poincaré index 0. Once the cone is passed, this unstable point disappears, and the homoclinic orbit emanating from it now becomes a periodic orbit, and the phase flow contains only two stable points, as it was at the beginning of the trip.

B. $P < 0$. Bifurcations through the tangent line

The direct transition from outside the cone to the inner part of the conoid (through its mutual line of tangency) is very special, since it consists of the evolution from a phase flow with only two equilibria to another with six equilibria. Let us describe how it happens.

For the sake of simplicity, we fix the negative value of P ($P = -2$, for instance); thus, we consider the evolution through the tangent point between the hypocycloid and the ellipse. The phase portrait is represented in Fig. 9.

In region A, there are only two equilibria on the equator ($w = 0$) that are stable, and the phase portrait is made of periodic orbits surrounding them. As the parameters approach the tangent point, one periodic orbit remains almost unaltered, whereas the periodic orbits close to it start to widen and distort, creating a kind of twofold womb where two new stable equilibria will be born. Indeed, just at the tangent point (in parameter plane), the unmodified orbit changes to be infinite periodic; the point of this orbit with $w = 0$ and $u > 0$ becomes a degenerate unstable equilibrium. As soon as the tangent point is crossed, this equilibrium splits into four equilibria, two unstable (E_0 and E_1), and two stable on the equator. We name this a *double teardrop* bifurcation.

VI. CONCLUSIONS

For the unique triparametric quadratic Hamiltonian on the unit sphere, a complete description of the parametric bifurcations and of the phase flow is presented. The bifurcation surfaces in the parametric space have been obtained. The biparametric case is recovered as a particular case of the triparametric one. A bifurcation termed the *double teardrop bifurcation* is reported.

ACKNOWLEDGMENTS

We are indebted to Professor Farrelly for his careful reading of the manuscript and for revising the English. This work has been supported in part by the Ministerio de Educación y Ciencia (DIGICYT Project No. PB94-0552-A) and by a grant of the Universidad de la Rioja (convenio Caja Rioja).

[1] T. Uzer *et al.*, Science **253**, 42 (1991); H. Friedrich, Phys. W **5**, 32 (1992).
[2] A.J. Lichtenberg and M.A. Lieberman, *Regular and Stochastic Motion*, Applied Mathematics Sciences Vol. 38 (Springer-Verlag, New York, 1983).
[3] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics Vol. 17 (Springer, New York, 1994).
[4] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer-Verlag, New York, 1989); *Dynamical Systems III* (Springer-Verlag, Berlin, 1988).
[5] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd ed. (Benjamin/Cummings, Reading, MA, 1980).
[6] A. Deprit and A. Elipe, J. Astron. Sci. **41**, 143 (1993).
[7] S. Coffey, A. Deprit, B. Miller, and C. Willians, Ann. N. Y. Acad. Sci. **497**, 2 (1987); D. Farrelly *et al.*, Phys. Rev. A **45**, 4738 (1992); J. Milligan and D. Farrelly, *ibid.* **47**, 3137 (1993); J. E. Howard and D. Farrelly, Phys. Lett. A **178**, 62 (1993); D. Farrelly and J. E. Howard, Phys. Rev. A **48**, 851 (1993); H. Friedrich and D. Wintgen, Phys. Rep. **183**, 37 (1989); S. Ferrer *et al.*, Phys. Lett. A **146**, 411 (1993); A. Elipe and S. Ferrer, in *Hamiltonian Dynamical Systems: History, Theory and Applications*, edited by H.S. Dumas, K.R. Meyer, and D.S. Schmidt IMA Series Vol. 63 (Springer-Verlag, New York, 1995), pp. 137–145.
[8] A. Elipe and S. Ferrer, Phys. Rev. Lett. **72**, 985 (1994).
[9] A. Deprit (unpublished).
[10] W. G. Harter and C. W. Patterson, J. Chem. Phys. **80**, 4241 (1984); M. E. Kellman and E. D. Lynch, *ibid.* **88**, 2205 (1988); L. Xiao and M. E. Kellman, *ibid.* **93**, 5805 (1990); M. E. Kellman and L. Xiao, *ibid.* **93**, 5821 (1990).
[11] D. David, D.D. Holm, and M.V Tratnik, Phys. Lett. A **137**, 355 (1989); **138**, 29 (1989).
[12] D. David, D.D. Holm, and M.V Tratnik, Phys. Rep. **187**, 281 (1990).
[13] S. Coffey, A. Deprit, and B. Miller, Celest. Mech. **39**, 365 (1986); S. Coffey, A. Deprit, and E. Deprit, Celest. Mech. Dynam. Astron. **59**, 37 (1994).
[14] S. Coffey, A. Deprit, E. Deprit, and L. Healy, Science **247**, 769 (1990).
[15] A. Deprit and A. Elipe, Celest. Mech. Dynam. Astron. **51**, 227 (1991); B. Miller, *ibid.* **51**, 251 (1991); A. Elipe and V. Lanchares, Bol. Astron. Obs. Madrid **12**, 56

(1990); A. Elipe, B. Miller and M. Vallejo, Astron. Astrophys. **300**, 722 (1995).
[16] A. Elipe and V. Lanchares, Mech. Res. Commun. **21**, 209 (1994).
[17] V. Lanchares and A. Elipe, Chaos **5**, 367 (1995).
[18] V. Lanchares and A. Elipe, Chaos (to be published).
[19] J. Frauendiener, Mech. Res. Commun. **22**, 313 (1995).
[20] D.J. Struik, *Lectures on Classical Differential Geometry*, 2nd ed. (Addison-Wesley, Reading, MA, 1961). Dover reprint.
[21] K. A. Ross, *Elementary Analysis: The Theory of Calculus* (Springer-Verlag, New York, 1980).
[22] A.A. Andronov, A.A. Vitt, and S.E. Khaikin, *Theory of Oscillators* (Dover, New York, 1966).
[23] L. Meirovitch, *Methods of Analytical Dynamics* (McGraw-Hill, New York, 1970).
[24] J.E. Howard and R.S. Mackay, Phys. Lett. A **122**, 331 (1987).
[25] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983).

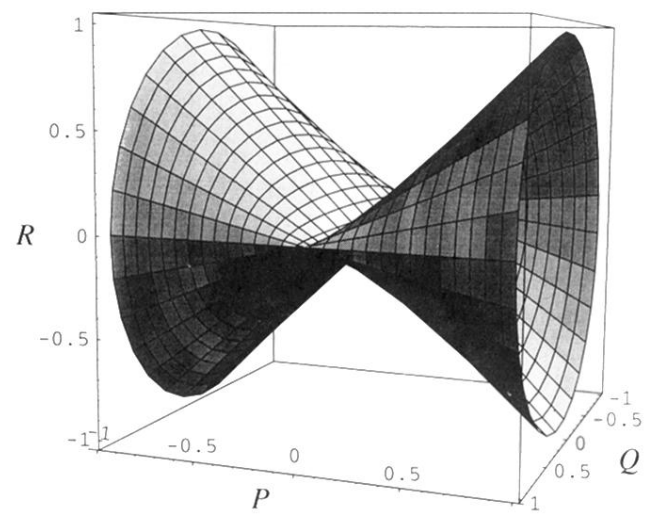

FIG. 1. Surface of bifurcation $\partial\mathcal{F} \equiv R^2 = P^2(1 - Q^2)$ for equilibria with $w \neq 0$.

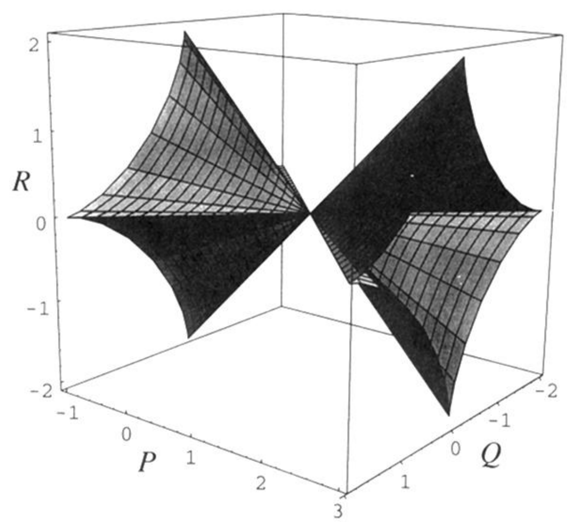

FIG. 2. Surface of bifurcation $\partial\mathcal{G} \equiv Q^{2/3} + R^{2/3} = (P-1)^{2/3}$ for equilibria with $w = 0$.

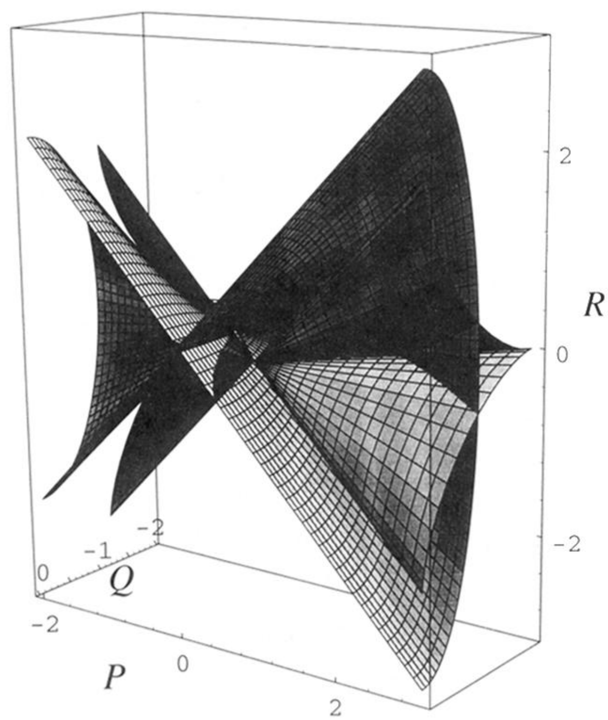

FIG. 3. The two surfaces of parametric bifurcations: the conoid $\partial\mathcal{F}$ and the cone $\partial\mathcal{G}$.